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Point symmetries of generalized Toda field theories

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Abstract. A class of two-dimensional field theories with exponential interactions is introduced. The interaction depends on two 'coupling' matrices and is sufficiently general to include all Toda field theories existing in the literature. Lie point symmetries of these theories are found for an infinite, semi-infinite and finite number of fields. Special attention is accorded to conformal invariance and its breaking.

1. Introduction

The purpose of this paper is to investigate the Lie point symmetries of a large class of 'generalized Toda field theories'. The class is characterized by the equation

$$u_{n,xy} = F_n \qquad F_n = \sum_{m=n-n_1}^{n+n_2} K_{nm} \exp\left(\sum_{l=m-n_3}^{m+n_4} H_{ml} u_l\right) \tag{1.1}$$

where K and H are some real constant matrices and $n_1, \ldots, n_4$ are some finite non-negative integers. The range of n may be infinite, semi-infinite or finite, hence the matrices K and H may also be infinite, semi-infinite or finite.

If the range of n is finite, K and H may be rectangular, not necessarily square. We assume that all the rows in H are different, that H contains no zero rows and K no zero columns. In all the cases we assume that the range of the interaction on the right-hand side of equation (1.1) is finite, hence the finite summation limits in both sums. 'Generalized Toda lattices' are obtained from equation (1.1) by symmetry reduction, using translational invariance, i.e. restricting to solutions of the form $u_n(x, y) = w_n(t)$ where $t = x + \lambda y$.

Toda lattices, their generalizations and Toda field theories represent one of the most interesting, rich and fruitful developments in the realm of completely integrable systems. The original Toda lattice was introduced by Toda [1, 2] who found analytical solitons and periodic solutions in a discrete lattice with an exponential potential involving nearest-neighbour interactions. It was also found that the Toda lattice admits a Lax representation and all the usual attributes of integrability [3, 4]. The Toda lattice was generalized to integrable lattices related to the root systems of simple Lie algebras [5–8]. The considered lattices can be finite, infinite, semi-infinite or periodic.

The attractive features of Toda lattices have been generalized to two space dimensions in several different ways [9–19].

All of them can be recovered from equation (1.1) by specifying the matrices K and H. Thus, the Mikhailov–Fordy–Gibbons field theories [9, 10] (for infinitely many fields)

$$u_{n,xy} = \mathrm{e}^{u_{n-1}-u_n} - \mathrm{e}^{u_n-u_{n+1}} \tag{1.2}$$

are obtained by putting $H_{nn-1} - H_{nn} = 1$, $K_{nn} = -K_{nn+1} = 1$ and all other components to zero. A class of Toda field theories

$$u_{n,xy} = \sum_{m=n-n_1}^{n+n_2} K_{nm}\mathrm{e}^{u_m} \tag{1.3}$$

studied by Leznov and Saveliev [12, 13], Olive, Turok and others [14–17] (usually for a finite number of fields u_n) are obtained by setting $H = I$ and taking K to be the Cartan matrix of a semisimple Lie algebra (or an affine one).

A further class of Toda field theories, also studied by Leznov and Saveliev [13, 14], Bilal and Gervais [17], and Babelon and Bonora [18] (for a finite number of fields) can be written as

$$u_{n,xy} = \exp \sum_{l=m-n_3}^{m+n_4} H_{nl}u_l \tag{1.4}$$

and is obtained by taking $K = I$ and H as a Cartan matrix.

In this paper we will be interested in point symmetries of the system (1.1), rather than in questions of integrability or explicit solutions. The symmetries we are interested in will include conformal invariance, whenever it is present, and gauge invariance, if not, however, higher, or generalized symmetries, be they local, or not.

In section 2 we consider infinite Toda field theories, i.e. take $-\infty < n < \infty$. In this case equation (1.1) can be viewed as a differential-difference equation. Continuous Lie symmetries of such equations have been studied using several different approaches [20–29]. We shall follow that of [20–24], using both the 'intrinsic method' and the 'differential equation method' [21].

In section 3 we turn to finite Toda field theories, when we have $1 \leqslant n \leqslant N < \infty$ in equation (1.1). Equation (1.1) in this case represents a system of N differential equations and its point symmetries can be obtained in a standard manner [30, 31]. We first obtain general results, and then specify the matrices H and K in several different ways.

Section 4 is devoted to semi-infinite Toda field theories, i.e. $0 \leqslant n < \infty$. Again we first obtain general results, and then specify the matrices H and K, enforcing the cut-off at $n = 0$ in several different ways.

Some conclusions are drawn in section 5.

2. Symmetries of generalized ∞-Toda field theories

2.1. General results

Let us consider equation (1.1) with n in the range $-\infty < n < \infty$. We follow the 'differential equation method' described in [21] and look for transformations of the form

$$\tilde{\vec{x}} = \Lambda_g(\vec{x}, \{u_k\}) \qquad \tilde{u}_n = \Omega_g(\vec{x}, n, \{u_k\}) \qquad \tilde{n} = n \tag{2.1}$$

where we have used the notation $\vec{x} \equiv (x, y)$, $\tilde{\vec{x}} \equiv (\tilde{x}, \tilde{y})$, taking solutions of equation (1.1) into solutions. The notation $\{u_k\}$ indicates that the new variables can depend on all the fields $\{u_k\}_{k\in\mathbb{Z}}$.

The Lie group transformation (2.1) is generated by a Lie algebra of vector fields of the form

$$\hat{v} = \xi(x, y, \{u_k\})\partial_x + \eta(x, y, \{u_k\})\partial_y + \sum_{j=-\infty}^{\infty} \phi_j(x, y, \{u_k\})\partial_{u_j}. \tag{2.2}$$

The prolongation of this vector field is constructed in the same manner as for differential equations [30, 31] (albeit an infinite system of them). For a general equation of the form

$$E_n = u_{n,xy} - F_n(x, y, \{u_k\}) = 0 \tag{2.3}$$

we require

$$pr^{(2)}\hat{v}E_n|_{E_n=0} = 0. \tag{2.4}$$

It was shown quite generally [21] that for equation (2.3) with F_n any sufficiently smooth function depending on at least one function u_k, $k \neq n$, the vector field (2.2) satisfying equation (2.4) will have the form

$$\xi = \xi(x) \qquad \eta = \eta(y) \qquad \phi_n = \sum_{k=-\infty}^{\infty} A_{nk}u_k + B_n(x, y) \tag{2.5}$$

where $A = \{A_{n\alpha}\}$ is a constant (infinite) matrix. The functions in equation (2.5) must satisfy a remaining determining equation, namely

$$B_{n,xy} - (\xi_x + \eta_y)F_n + \sum_{\alpha=-\infty}^{\infty} A_{n\alpha}F_\alpha - \xi F_{n,x} - \eta F_{n,y} - \sum_{\alpha=-\infty}^{\infty}\left(\sum_{\beta=-\infty}^{\infty} A_{\alpha\beta}u_\beta + B_\alpha\right)F_{n,u_\alpha} = 0 \tag{2.6}$$

where F_{n,u_α} is the derivative of F_n with respect to the variable u_α.

Let us now specify the function F_n to be a sum of exponentials as in equation (1.1). There are three types of terms in equation (2.6): those independent of u_n, linear in u_n times exponentials and pure exponentials. Each type of term must vanish separately. Since H has no zero rows we obtain the determining equations

$$B_{n,xy} = 0 \tag{2.7}$$

$$\sum_{\alpha=-\infty}^{\infty} A_{\alpha m}F_{n,u_\alpha} = 0 \tag{2.8}$$

$$-(\xi_x + \eta_y)F_n + \sum_{\alpha=-\infty}^{\infty} A_{n\alpha}F_\alpha - \sum_{\alpha=-\infty}^{\infty} B_\alpha F_{n,u_\alpha} = 0. \tag{2.9}$$

Equation (2.8) can be rewritten as

$$\sum_{\alpha\beta} K_{n\beta}H_{\beta\alpha}A_{\alpha m}\exp\left(\sum_\gamma H_{\beta\gamma}u_\gamma\right) = 0. \tag{2.10}$$

All exponentials in equation (2.10) are linearly independent (since all rows in H are different), so the equation must hold for each β separately and the exponentials can be dropped. Moreover, the factor $K_{n\beta}$ can be dropped (since K has no zero column). We find that equation (2.8) in this case implies an equation for the matrix A, namely

$$\sum_{\alpha=-\infty}^{\infty} H_{n\alpha} A_{\alpha m} = 0 \tag{2.11}$$

or in matrix form $HA = 0$ (however, the matrices are infinite).

Let us now turn to equation (2.9) and make use of the finite range of the interaction F_n in equation (1.1). We have

$$\frac{\partial F_n}{\partial u_k} = 0 \qquad n + n_u < k \quad \text{or} \quad k < n - n_d \tag{2.12}$$

for some non-negative integers n_u and n_d. In equation (2.9) all exponentials, obtained after substituting for F_n from equation (1.1), are linearly independent. This allows us to split equation (2.9) into two types of equations. These are obtained as coefficients of $\exp\left(\sum_l H_{ml} u_l\right)$, with $m \in [n - n_1, n + n_2]$ and with m outside this interval, respectively. Thus we have

$$-K_{nm}\left[(\xi_x + \eta_y) + \sum_{\alpha=m-n_3}^{m+n_4} B_\alpha H_{m\alpha}\right] + \sum_{\rho=m-n_1}^{m+n_2} A_{n\rho} K_{\rho m} = 0 \qquad m \in [n - n_1, n + n_2] \tag{2.13}$$

$$\sum_{\rho=m-n_1}^{m+n_2} A_{n\rho} K_{\rho m} = 0 \qquad m \notin [n - n_1, n + n_2]. \tag{2.14}$$

We shall show that equation (2.14) actually holds for all values of m so that equation (2.13) can be simplified. To do this, we view equation (2.11) as a difference equation for $A_{\alpha m}$. To make this explicit we restrict H and K to be band matrices, with finite bands of constant width

$$H_{nm} = H_{n,n+\sigma} = \begin{cases} h_\sigma(n) & \sigma \in [p_1, p_2] \\ 0 & \sigma \notin [p_1, p_2] \end{cases} \qquad h_{p_1}(n) \neq 0 \qquad h_{p_2}(n) \neq 0. \tag{2.15}$$

Similarly,

$$K_{nm} = K_{m+\sigma,m} = \begin{cases} k_\sigma(m) & \sigma \in [q_1, q_2] \\ 0 & \sigma \notin [q_1, q_2] \end{cases} \qquad k_{q_1}(m) \neq 0 \qquad k_{q_2}(m) \neq 0. \tag{2.16}$$

In these notation we see that equation (2.11) is a linear difference equation for $A_{\sigma m}$ with $p_1 - p_2 + 1$ terms,

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n) A_{\sigma+n,m} = 0. \tag{2.17}$$

Equation (2.17) determines the dependence of A_{nm} on n. Indeed, the linear difference equation

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\, \psi_{\sigma+n} = 0 \tag{2.18}$$

has $p_2 - p_1$ linearly independent solutions, a basis of which we denote by $\left\{\psi_n^j, j = 1, 2, \ldots, p_2 - p_1\right\}$. Thus, we have

$$A_{nm} = \sum_{j=1}^{p_2-p_1} \psi_n^j C_{jm} \tag{2.19}$$

where C_{jm} are constants to be determined by the remaining determining equations (2.13) and (2.14). In order to analyse them, let us define the quantities

$$Q_{nm} = \sum_{\sigma=m-n_1}^{m+n_2} A_{n\sigma} K_{\sigma m}.$$

From equation (2.14) we have $Q_{nm} = 0$ for m 'sufficiently far away' from n. However, by using the expansion (2.19), we obtain

$$Q_{nm} = \sum_{j=1}^{p_2-p_1} \psi_n^j \sum_{\sigma=m-n_1}^{m+n_2} C_{j\sigma} K_{\sigma m}$$

which, because of the linear independence of the ψ_n^j, implies

$$\sum_{\sigma=m-n_1}^{m+n_2} C_{j\sigma} K_{\sigma m} = 0 \tag{2.20}$$

for all values of m, since this relation does not depend on n and the index m is no longer tied to n. In other words, if $Q_{nm} = 0$ holds for certain values of n and m, as in equation (2.14), then that equation must hold for all values. As in the case of equation (2.17), we introduce a solution basis $\left\{\phi_m^l, l = 1, \ldots, q_2 - q_1\right\}$ for the equation

$$\sum_{\sigma=q_1}^{q_2} k_\sigma(m)\, \phi_{\sigma+m} = 0. \tag{2.21}$$

The general solution of equation (2.20) is now

$$C_{jm} = \sum_{l=1}^{q_2-q_1} q_{jl}\, \phi_m^l$$

where q_{jl} are arbitrary constants. The expression (2.19) for A_{nm} is replaced by

$$A_{nm} = \sum_{j=1}^{p_2-p_1} \sum_{l=1}^{q_2-q_1} q_{jl}\, \psi_n^j \phi_m^l. \tag{2.22}$$

A further consequence is that the last term in equation (2.13) can be dropped. Then, using the general solution for equation (2.7)

$$B_n(x, y) = \beta_n(x) + \gamma_n(y)$$

we separate the x from the y dependence in equation (2.13) and reduce it to two inhomogeneous difference equations for $\beta_n(x)$ and $\gamma_n(y)$. The general solutions of which are

$$\beta_n(x) = \sum_{j=1}^{p_2-p_1} r_j(x)\, \psi_n^j - b_n \xi_x(x) \qquad \gamma_n(x) = \sum_{j=1}^{p_2-p_1} s_j(y)\, \psi_n^j - b_n \eta_y(y) \tag{2.23}$$

where b_n is an arbitrarily chosen solution of the inhomogeneous difference equation

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\, b_{\sigma+n} = 1. \tag{2.24}$$

Furthermore, in equation (2.23) the functions $r_j(x)$ and $s_j(y)$ are chosen arbitrarily. Finally, we obtain the following theorem.

Theorem 1. *Consider all the generalized Toda theories of the form (1.1) for infinitely many fields $u_n(x,y)$, where the coupling matrices H and K satisfy equations (2.15) and (2.16). Their Lie point symmetry algebra is infinite dimensional and a basis for it is given by the following vector fields:*

$$\hat{X}(\xi) = \xi(x)\partial_x - \xi_x(x)\sum_{n=-\infty}^{\infty} b_n\partial_{u_n} \qquad \hat{Y}(\eta) = \eta(y)\partial_y - \eta_y(y)\sum_{n=-\infty}^{\infty} b_n\partial_{u_n} \tag{2.25}$$

$$\hat{U}_j(r_j) = r_j(x)\sum_{n=-\infty}^{\infty}\psi_n^j\partial_{u_n} \qquad \hat{V}_j(s_j) = s_j(y)\sum_{n=-\infty}^{\infty}\psi_n^j\partial_{u_n} \qquad (j = 1,\ldots,p_2-p_1) \tag{2.26}$$

$$\hat{Z}_{jl} = \left(\sum_{m=-\infty}^{\infty}\phi_m^l u_m\right)\left(\sum_{n=-\infty}^{\infty}\psi_n^j\partial_{u_n}\right) \qquad (j = 1,\ldots,p_2-p_1;\ l = 1,\ldots,q_2-q_1). \tag{2.27}$$

The functions $\xi(x)$, $\eta(y)$, $r_j(x)$ and $s_j(y)$ are arbitrary, all the other quantities are determined by solving the linear difference equations (2.18), (2.21) and (2.24).

As far as interpretation is concerned, we see that the generalized ∞-Toda lattice (1.1) is always conformally invariant, since the vector fields (2.25) generate arbitrary reparametrizations of x and y, accompanied by appropriate transformations of the fields u_n. More specifically, the conformal transformations leaving equation (1.1) invariant are

$$\begin{aligned}\tilde{x} &= \psi(x,\lambda) \qquad \tilde{y} = \chi(y,\lambda)\\ \tilde{u}_n(\tilde{x},\tilde{y}) &= u_n(x,y) - b_n\ln\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\frac{\mathrm{d}\chi}{\mathrm{d}y}\right)\end{aligned} \tag{2.28}$$

where $\psi(x,\lambda)$ and $\chi(y,\lambda)$ are arbitrary functions of x and y, related to $\xi(x)$ and $\eta(y)$ by the relations

$$\begin{aligned}\tilde{x} &= \psi(x,\lambda) = T^{-1}(\lambda + T(x))\\ \tilde{y} &= \chi(y,\lambda) = S^{-1}(\lambda + S(y))\end{aligned} \tag{2.29}$$

with

$$T(x) = \int_0^x \frac{\mathrm{d}s}{\xi(s)} \qquad S(y) = \int_0^y \frac{\mathrm{d}t}{\eta(t)}. \tag{2.30}$$

The vector fields $\hat{U}_j(r)$ and $\hat{V}_j(s)$ generate gauge transformations: certain functions obtained by integrating the vector fields can be added to any solution. Formally, the operators $\hat{Z}_{jl}$ generate linear transformations among components of solutions. However, the sums are over an infinite range, so convergence problems may arise. Moreover, we have

$$\partial_{xy}\left(\sum_m \phi_m^l u_m\right) = 0 \tag{2.31}$$

as a consequence of equation (2.21). In other words, if equation (2.21) admits non-trivial solutions, than one can always perform a linear transformation among the u_n's, in such a way that $q_2 - q_1$ new fields $v_l = \sum_m \phi_m^l u_m$, satisfying the wave equation $\partial_x\partial_y v_l = 0$, are replaced in the Toda system.

As stated in theorem 1, the problem of finding all symmetries of equation (1.1) reduces to solving the recursion relations (2.18), (2.21) and (2.24). In general, this may not be possible analytically in closed form. Well developed techniques exist for solving homogeneous and inhomogeneous difference equations with constant coefficients [32, 33]. This is the case that occurs for all generalized Toda field theories that we found in the literature: $h_\sigma(n)$ and $k_\sigma(m)$ do not depend on n and m, respectively. The non-zero commutation relations for the symmetry algebra of the generalized ∞-Toda theory (1.1) are:

$$\begin{aligned}
&[\hat{X}(\xi_1), \hat{X}(\xi_2)] = \hat{X}(\xi_1\,\xi_{2,x} - \xi_{1,x}\,\xi_2) && [\hat{Y}(\eta_1), \hat{Y}(\eta_2)] = \hat{Y}(\eta_1\,\eta_{2,y} - \eta_{1,y}\,\eta_2)\\
&[\hat{X}(\xi), \hat{U}_j(r)] = \hat{U}_j(\xi r_x) && [\hat{Y}(\eta), \hat{V}_j(s)] = \hat{V}_j(\eta s_y)\\
&[\hat{X}(\xi), \hat{Z}_{jl}] = -\hat{U}_j\left(\xi_x\sum_n b_n\phi^l_n\right) && [\hat{Y}(\eta), \hat{Z}_{jl}] = -\hat{V}_j\left(\eta_y\sum_n b_n\phi^l_n\right)\\
&[\hat{U}_a(r), \hat{Z}_{jl}] = \hat{U}_j\left(r\sum_m \phi^l_m\psi^a_m\right) && [\hat{V}_a(s), \hat{Z}_{jl}] = \hat{V}_j\left(s\sum_m \phi^l_m\psi^a_m\right)\\
&[\hat{Z}_{ab}, \hat{Z}_{cd}] = \left(\sum_m \phi^d_m\psi^a_m\right)\hat{Z}_{cb} - \left(\sum_m \phi^b_m\psi^c_m\right)\hat{Z}_{ad}.
\end{aligned} \tag{2.32}$$

The algebra of vector fields $\hat{Z}_{jl}$ is finite dimensional (its dimension is $d = (p_2-p_1)\times(q_2-q_1)$). However, its isomorphism class cannot be determined without specifying the functions ϕ^l_m and ψ^j_n, i.e. the matrices H and K in (1.1). In all examples in the literature, we have either $d = 1$ or 0. It is, however, easy to invent examples in which $\{\hat{Z}_{jl}\}$ is simple, semisimple, solvable or whatever we postulate *a priori*.

The overall structure of the obtained Lie algebra is

$$(\{\hat{X}\} \oplus \{\hat{Y}\}) \niplus (\{\hat{Z}\} \niplus (\hat{U} \oplus \hat{V})). \tag{2.33}$$

If $\{\hat{Z}\}$ is solvable, then (2.33) amounts to a Levi decomposition, since both $\{\hat{X}\}$ and $\{\hat{Y}\}$ are centreless Virasoro algebras and hence simple. We recall that the Levi theorem does not hold for infinite-dimensional Lie algebras and a Levi decomposition does not necessarily exist.

Let us sum up the general results obtained so far for the symmetries of the generalized ∞-Toda field theories (1.1) under the constraints imposed in theorem 1.

(a) The theory is always conformally invariant, since the inhomogeneous equation (2.24) always has a solution.
(b) The theory allows gauge transformations $\hat{U}$ and $\hat{V}$ if $p_2 - p_1 \geqslant 1$.
(c) The transformations of type $\hat{Z}$ exist if $(p_2 - p_1)(q_2 - q_1) \geqslant 1$.

2.2. Special cases

2.2.1. The Mikhailov–Fordy–Gibbons two-dimensional ∞-Toda system (1.2). We have

$$h_{-1}(n) = -h_0(n) = 1 \qquad \text{and} \qquad k_{-1}(n) = -k_0(n) = -1 \tag{2.34}$$

so $p_2 - p_1 = q_2 - q_1 = 1$. From equations (2.18) and (2.21) we have

$$\psi_m = \phi_m = 1.$$

Equations (2.23) and (2.24) in this case imply

$$\beta_n = \beta(x) + n\xi_x \qquad \gamma_n = \gamma(y) + n\eta_y.$$

From theorem 1 we now obtain all symmetries of equation (1.2), namely

$$\begin{aligned}
&\hat{X}(\xi) = \xi(x)\partial_x + \xi_x \sum_{n=-\infty}^{\infty} n\partial_{u_n} \qquad &&\hat{Y}(\eta) = \eta(y)\partial_y + \eta_y \sum_{n=-\infty}^{\infty} n\partial_{u_n} \\
&\hat{U} = \beta(x) \sum_{n=-\infty}^{\infty} \partial_{u_n} &&\hat{V} = \gamma(y) \sum_{n=-\infty}^{\infty} \partial_{u_n} \\
&\hat{Z} = \left(\sum_{m=-\infty}^{\infty} u_m\right)\left(\sum_{n=-\infty}^{\infty} \partial_{u_n}\right).
\end{aligned} \tag{2.35}$$

The generators $\hat{X}$, $\hat{Y}$, $\hat{U}$ and $\hat{V}$ were obtained in [21] using the so-called 'intrinsic method'. The generator $\hat{Z}$ was not obtained there and cannot be obtained by the intrinsic method.

2.2.2. The Toda field theory (1.3). We take $H = I$. Then equations (2.18), (2.21) and (2.24) in this case imply

$$\beta_m = -\xi_x \qquad \gamma_m = -\eta_y \qquad A_{nm} = 0.$$

The theory is only conformally invariant

$$\hat{X}(\xi) = \xi(x)\partial_x - \xi_x \sum_n \partial_{u_n} \qquad \hat{Y}(\eta) = \eta(y)\partial_y - \eta_y \sum_n \partial_{u_n} \tag{2.36}$$

and no further symmetries are obtained.

2.2.3. The Toda field theories (1.4). We take $K = I$ and relation (2.21) implies

$$A_{nm} = 0.$$

The remaining equations (2.24) cannot be solved explicitly for general $h_\sigma(m)$, but as mentioned above, we can easily deal with in the constant coefficients case. As an example, let us restrict to the case when H is the A_∞ Cartan matrix (this is the A_N Cartan matrix for $N \to \infty$, where the limit is taken symmetrically from a fixed, but not extremal, vertex in the corresponding Dynkin diagram). Thus we have

$$h_{-1} = h_{+1} = -1 \qquad h_0 = 2 \tag{2.37}$$

the solutions (2.23) become

$$\beta_n = \tfrac{1}{2}n^2\,\xi_x + n\,r_2(x) + r_1(x) \qquad \gamma_n = \tfrac{1}{2}n^2\,\eta_y + n\,s_2(y) + s_1(y). \tag{2.38}$$

The symmetry algebra is

$$\begin{aligned}
&\hat{X}(\xi) = \xi(x)\partial_x + \tfrac{1}{2}\xi_x \sum_{n=-\infty}^{\infty} n^2\partial_{u_n} \qquad &&\hat{Y}(\eta) = \eta(y)\partial_y + \tfrac{1}{2}\eta_y \sum_{n=-\infty}^{\infty} n^2\partial_{u_n} \\
&\hat{U}_1(r_1) = r_1(x) \sum_{n=-\infty}^{\infty} \partial_{u_n} &&\hat{V}_1(s_1) = s_1(y) \sum_{n=-\infty}^{\infty} \partial_{u_n} \\
&\hat{U}_2(r_2) = r_2(x) \sum_{n=-\infty}^{\infty} n\,\partial_{u_n} &&\hat{V}_2(s_2) = s_2(y) \sum_{n=-\infty}^{\infty} n\partial_{u_n}
\end{aligned} \tag{2.39}$$

where $\xi(x)$, $\eta(y)$, $r_1(x)$, $s_1(y)$, $r_2(x)$ and $s_2(y)$ are arbitrary smooth functions.

3. Symmetries of finite generalized Toda field theories

3.1. General results

In this case we have a system of N partial differential equations in N fields $u_n(x, y)$, namely

$$u_{n,xy} = F_n \qquad F_n = \sum_{m=1}^{M} K_{nm} \exp\left(\sum_{l=1}^{N} H_{ml} u_l\right) \qquad (1 \leqslant n \leqslant N). \tag{3.1}$$

The 'coupling constant' matrices H and K satisfy $H \in \mathbb{R}^{M\times N}$ and $K \in \mathbb{R}^{N\times M}$. The system (3.1) could arise in a quite general field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{m,n=1}^{N} \kappa_{mn} \partial_x u_m \partial_y u_n - \sum_{m=1}^{M} c_m \exp\left(\sum_{l=1}^{N} H_{ml} u_l\right) \qquad (c_m \neq 0) \tag{3.2}$$

with

$$K = L^{-1} H^T C \qquad L = \frac{1}{2}(\kappa + \kappa^T) \qquad C = \mathrm{diag}(c_1, \ldots, c_N). \tag{3.3}$$

Some general considerations concerning the system (3.1) are in order.

First, if either K or H (or both) allow an inverse, or at least a left inverse, then this system can be simplified. Indeed, let K^{-1} exist. We put $u_n = \sum_m K_{nm} \rho_m$ and obtain

$$\rho_{m,xy} = \exp\left(\sum_{l=1}^{M} (HK)_{ml} \rho_l\right) \qquad 1 \leqslant m \leqslant M. \tag{3.4}$$

Conversely, let H^{-1} exist and put $w_j = \sum_l H_{jl} u_l$, we obtain

$$w_{m,xy} = \sum_{j=1}^{M} (HK)_{mj} \mathrm{e}^{w_j} \qquad 1 \leqslant m \leqslant M. \tag{3.5}$$

In other words, one of the matrices H or K can be normalized to I_M, if it is left invertible.

The second comment is that the system (3.1) with $K = I$ admits Lie–Bäcklund transformations, and in this sense is completely integrable, if the matrix H is a Cartan, or a generalized Cartan matrix [19].

We mention that in the case of the infinite Toda field theories the matrices H and K in general have non-trivial kernels, are hence not invertible and we cannot normalize them.

Let us now turn to the Lie point symmetries of the system (3.1). We write a general element of the symmetry algebra in the form (2.2) (with the sum in the range $1 \leqslant n \leqslant N$), apply its prolongation to equation (3.1) as in equation (2.4). From the determining equations we find that for any F_n in equation (3.1), in complete analogy with the ∞-Toda theory, a general element of the symmetry algebra will have the form (2.5), the summation being from 1 to N.

Two determining equations remain and they depend on the specific form of F_n in equation (3.1). Making use of the fact that all the exponentials are linearly independent (no two rows in H coincide) and that the matrix K has no zero column, we reduce the remaining determining equations to two matrix relations

$$HA = 0 \tag{3.6}$$

$$[(A - (\xi_x + \eta_y)I)K]_{nm} = K_{nm}(HB)_m \qquad (1 \leqslant n \leqslant N, 1 \leqslant m \leqslant M). \tag{3.7}$$

We multiply equation (3.7) by H from the left and use (3.6) to obtain

$$-(\xi_x + \eta_y)(HK)_{km} = (HK)_{km}(HB)_m \qquad \forall k, m. \tag{3.8}$$

If the matrix HK has no zero column, then we obtain

$$HB = -(\xi_x + \eta_y)\bar{\mathbf{1}}_M \tag{3.9}$$

where $\bar{\mathbf{1}}_M = (1, \ldots, 1)^T \in \mathbb{R}^M$, and from equation (3.7)

$$AK = 0. \tag{3.10}$$

Thus, matrix A must satisfy the same two homogeneous equations (3.6) and (3.10) as in the infinite case. Furthermore, if $\bar{\mathbf{1}}_M$ is in the image of H, then we define $\boldsymbol{b}_N \in \mathbb{R}^N$ to be an arbitrarily chosen (but specified) solution of the inhomogeneous equation

$$H\boldsymbol{b}_N = \bar{\mathbf{1}}_M. \tag{3.11}$$

The results of these considerations can be summed up as follows.

Theorem 2. *Consider the generalized Toda field theories (3.1) with a finite number of fields N. Assume that all rows in H are different and that the matrix $H\ K$ has no zero column. Then three types of symmetries can occur and they depend on the properties of the fundamental spaces of the matrices H and K. The symmetries are of the same form as in theorem 1, except that all summations range from 1 to N. However, if $\bar{\mathbf{1}}_M \in \mathrm{Im}(H)$, then ξ and η are arbitrary functions of x and y, respectively, and the theory is conformally invariant. The quantities b_n are the components of the vector $\boldsymbol{b}_N$, itself an arbitrary solution of equation (3.11). Otherwise, if $\bar{\mathbf{1}}_M \notin \mathrm{Im}(H)$, the theory is invariant only under the Poincaré group, generated by*

$$\hat{P}_1 = \partial_x \qquad \hat{P}_2 = \partial_y \qquad \hat{L} = x\partial_x - y\partial_y. \tag{3.12}$$

Gauge transformations exist only if H is not invertible. Analogously to the formulae (2.26), r_j and s_j are arbitrary functions and the vectors ψ^j span $\mathrm{Ker}(H)$. Finally, the vectors ϕ^l span the left kernel of K. If this space is not zero, then $\dim\left(\mathrm{Ker}\left(K^T\right)\right) \times \dim(\mathrm{Ker}(H))$ symmetries of the form (2.27) are admitted.

From theorem 2, in contrast to the case of infinitely many fields, conformal invariance is not *a priori* guaranteed, but it imposes restrictions on the image of H. Gauge symmetries exist only if the matrix H has a non-zero kernel.

3.2. Special cases

3.2.1. The Mikhailov–Fordy–Gibbons Toda theory and generalizations. Consider the field equation

$$\boldsymbol{U}_{xy} = \frac{\mu^2}{\beta}\sum_{i=1}^{N}\frac{\alpha_i}{\alpha_i^2}\exp(\beta\alpha_i\cdot\boldsymbol{U}) \tag{3.13}$$

where $\boldsymbol{U} = (u_1, \ldots, u_N)$ is an N-ple of real fields and $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ denote the simple roots of a classical simple finite Lie algebra. Equations (3.13) above take the form (1.2) for all n satisfying $N_0 \leqslant n \leqslant N-1$. For $n = N$ we obtain

$$u_{N,xy} = \exp(u_{N-1} - u_N). \tag{3.14}$$

The equations for $1 \leqslant n < N_0$ are different for each Cartan series. The number N_0 is equal to 2 for A_N, B_N, C_N, and 3 for D_N.

For the A_N algebra we have

$$u_{1,xy} = -\exp(u_1 - u_2). \tag{3.15}$$

Conformal and gauge transformations are exactly the same as given in equation (2.35) (except that the summations are from 1 to N).

For the B_N algebra we have

$$u_{1,xy} = \exp(-u_1) - \exp(u_1 - u_2). \tag{3.16}$$

Conformal transformations are as in equation (2.35) (with the same comment about the summations) and there is no gauge invariance.

For the C_N algebra we have

$$u_{1,xy} = -\exp(u_1 - u_2) + 2\exp(-2u_1). \tag{3.17}$$

The only symmetry is conformal invariance, generated by

$$\begin{aligned} \hat{X}(\xi) &= \xi(x)\partial_x + \xi_x \sum_{n=1}^{N} \left(n - \tfrac{1}{2}\right) \partial_{u_n} \\ \hat{Y}(\eta) &= \eta(y)\partial_y + \eta_y \sum_{n=1}^{N} \left(n - \tfrac{1}{2}\right) \partial_{u_n}. \end{aligned} \tag{3.18}$$

Finally, for the D_N algebra we have

$$\begin{aligned} u_{1,xy} &= \exp(-u_1 - u_2) - \exp(u_1 - u_2) \\ u_{2,xy} &= \exp(-u_1 - u_2) + \exp(u_1 - u_2) - \exp(u_2 - u_3). \end{aligned} \tag{3.19}$$

Again, the only symmetry is conformal invariance, in this case generated by

$$\begin{aligned} \hat{X}(\xi) &= \xi(x)\partial_x + \xi_x \sum_{n=1}^{N} (n-1)\partial_{u_n} \\ \hat{Y}(\eta) &= \eta(y)\partial_y + \eta_y \sum_{n=1}^{N} (n-1)\partial_{u_n}. \end{aligned} \tag{3.20}$$

We mention that the infinite system (1.2) can also be reduced to the finite one by imposing periodicity $u_{N+1} = u_1$. In this case $\bar{\mathbf{1}}_N$ is not contained in $\mathrm{Im}(H)$ and there is no conformal invariance. Thus, the symmetry is given by the two-dimensional Poincaré algebra (3.12) and by the gauge generators given in (2.35).

3.2.2. The Toda field theory (1.3). The symmetries are the same in the finite case as in the infinite one, namely the conformal transformations generated by (2.36) (for any finite matrix k).

3.2.3. The finite Toda theories (1.4). Since the Cartan matrix H is invertible, this theory is equivalent to that described by equation (1.3) in the sense of equations (3.4) and (3.5). Hence this theory is always and only conformally invariant. However, the generators of the vector fields take a slightly different form, which we report for a subsequent discussion.

For the A_N algebra the generators are given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y + \tfrac{1}{2}(\xi_x + \eta_y) \sum_{n=1}^{N} n(n - N - 1)\partial_{u_n}. \tag{3.21}$$

For the B_N algebra, the symmetry generator is given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y - \tfrac{1}{4}(\xi_x + \eta_y)\left\{N(N+1)\partial_{u_1} + 2\sum_{n=2}^{N}[N(N+1) - n(n-1)]\partial_{u_n}\right\}. \tag{3.22}$$

For the C_N algebra, the symmetry generator is given by

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y + \tfrac{1}{2}(\xi_x + \eta_y)\sum_{n=1}^{N}\left[n(n-2) - N^2 + 1\right]\partial_{u_n}. \tag{3.23}$$

Finally, for the D_N algebra ($N \geqslant 4$), one has

$$\hat{W} = \xi(x)\partial_x + \eta(y)\partial_y - \tfrac{1}{4}(\xi_x + \eta_y)\left\{N(N-1)(\partial_{u_1} + \partial_{u_2}) + 2\sum_{n=3}^{N}[N(N-1) - (n-2)(n-1)]\partial_{u_n}\right\}. \tag{3.24}$$

4. Symmetries of generalized semi-infinite Toda field theories

4.1. General results

Let us now restrict the range of the discrete variable n to be $1 \leqslant n < \infty$. Both the equations (1.1) of the generalized Toda field theories, and their symmetries will be modified. The matrices H and K will no longer be pure band matrices but will have the form

$$H = \begin{pmatrix} H_{1,1} & \ldots & \ldots & \ldots & H_{1,N} & & & & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & & & & & & & \\ H_{M,1} & \ldots & \ldots & \ldots & H_{M,N} & & & & & & & \\ & H_{M+1,M+1+p_1} & & \ldots & \ldots & \ldots & \ldots & H_{M+1,M+1+p_2} & & & & \\ & & & & H_{M+2,M+2+p_1} & \ldots & \ldots & \ldots & \ldots & \ldots & H_{M+2,M+2+p_2} & \\ & & & & & \ddots & \ddots & & \ddots & & \ddots & \ddots \end{pmatrix} \tag{4.1}$$

where $M + p_1 \leqslant N \leqslant M + p_2$ and the void entries are equal to zero. Similarly, the matrix K takes the form

$$K = \begin{pmatrix} K_{1,1} & \ldots & K_{1,N'} & & & \\ \ldots & \ldots & \ldots & K_{N'+1+q_1,N'+1} & & \\ K_{M',1} & \ldots & K_{M',N'} & \ldots & K_{N'+2+q_1,N'+2} & \\ & & & \ldots & \ldots & \ddots \\ & & & K_{N'+1+q_2,N'+1} & \ldots & \ddots \\ & & & & K_{N'+2+q_2,N'+2} & \ddots \\ & & & & & \ddots \end{pmatrix} \tag{4.2}$$

where $N'+q_1 \leqslant M' \leqslant N'+q_2$. Although one could easily construct non-trivial models, which do not fit in the given scheme, they seem quite artificial and, moreover, all the cases which we found in the literature satisfy the above restrictions.

We denote by $\tilde{H}$ and $\tilde{K}$, respectively, the $M \times N$ and $M' \times N'$ matrices, which can be extracted by taking the first M rows and the first N columns from H and, in turn, the first M' rows and the first N' columns from K.

The symmetry algebra of the semi-infinite Toda field theory equation can either be obtained directly, *ab initio*, or we can obtain it from the infinite case of section 2, by adding appropriate boundary conditions and requiring that they be invariant. As above, the functions $\xi(x)$, $\eta(y)$, A_{mn} and $B_n(x, y)$ must satisfy the remaining determining equations (2.7)–(2.9). Following the same reasoning as in the finite case (see section 3), we obtain the analogues of all the relations (3.6)–(3.10), where now all the labels and summations range from 1 to ∞ (i.e. we take $N \to \infty$ in all formulae). The key equation of the discussion is equation (3.9) and its associated homogeneous system. Here, we separate the problem into the finite subsystems

$$\tilde{H}\,\tilde{\boldsymbol{B}} = 0 \tag{4.3}$$

$$\tilde{H}\,\tilde{\boldsymbol{B}} = -(\xi_x + \eta_y)\bar{\mathbf{1}}_M \tag{4.4}$$

where $\tilde{\boldsymbol{B}} = (B_1, \ldots, B_N)$, and a difference linear equation, which we can put again in the form (2.18) or (2.24), respectively, for $n \geqslant M+1$. Equation (4.3) has $\mathrm{Ker}(\tilde{H})$ as its solution space. On the other hand, the difference equation (2.18) has a $(p_2 - p_1)$-dimensional solution space, the elements of which have the form

$$B_n = \sum_{j=1}^{p_2-p_1} \alpha_j \psi_n^j \qquad n \geqslant M+1+p_1 \tag{4.5}$$

in terms of the basis $\{\psi_n^j\}$. Moreover, the difference equation (2.18) has only the zero solution in the case $p_1 = p_2$. However, because of the imposed restrictions on the form of H, in such a case the components of the vector $\tilde{\boldsymbol{B}}$ are decoupled from the remaining $(B_{N+1}, \ldots)$. This means that the semi-infinite homogeneous linear system $HB = 0$ has a zero-dimensional kernel only if both the finite system (4.3) and the homogeneous difference equation (2.18) do.

Assuming now that $p_1 < p_2$ and, moreover, that $M + p_1 + 1 \leqslant N$, the components $(B_{M+1+p_1}, \ldots, B_N)$ have to satisfy both the finite linear equation (4.3) and the difference equation (2.18). Substituting the representation (4.5) into (4.3), we obtain $N - \dim(\mathrm{Ker}(\tilde{H}))$ constraints on the $\{\alpha_i\}_{i=1,\ldots,p_2-p_1}$. Thus, if it results that

$$M - N + p_2 + \dim(\mathrm{Ker}(\tilde{H})) = n_0 > 0 \tag{4.6}$$

then the semi-infinite homogeneous system $HB = 0$ admits an n_0-dimensional kernel, spanned by the set of linearly independent functions $\{\chi_n^j\}_{j=1,\ldots,n_0}$.

The above result implies that, if the constraint (4.6) holds, then the semi-infinite Toda model defined by (4.1) and (4.2) possesses a symmetry group of gauge transformations, generated by the $2 \times n_0$ vector fields

$$\hat{U}_j(r_j) = r_j(x)\sum_{n=1}^{\infty}\chi_n^j\partial_{u_n} \qquad \hat{V}_j(s_j) = s_j(y)\sum_{n=1}^{\infty}\chi_n^j\partial_{u_n} \qquad (j = 1, \ldots, p_2 - p_1). \tag{4.7}$$

As in the finite case, a semi-infinite theory is conformally invariant if the inhomogeneous equation (3.9) (for semi-infinite matrices) has a solution. Thus, now we must require that the vector $\bar{\mathbf{1}} = (1, 1, \ldots)$ be contained in $\mathrm{Im}(H)$. However, as outlined above, the problem is reduced to finding a solution of the equation (4.4) and of the difference equation (2.24). The former equation is solved if

$$\bar{\mathbf{1}}_M \in \mathrm{Im}(\tilde{H}). \tag{4.8}$$

For the difference equation (2.24) a solution always exists as seen in section 2. Hence the structure of the matrix H shown in (4.1) guarantees that a solution of the total inhomogeneous system always exists, once equation (4.8) is satisfied here. In conclusion, the condition (4.8) is not only necessary, but also sufficient to ensure the conformal invariance of the given Toda theories.

Finally, an analysis similar to the study of the gauge invariance can be performed for the $\hat{Z}$-type transformations, which exist if a common solution of the two semi-infinite homogeneous systems

$$HA = 0 \qquad AK = 0 \tag{4.9}$$

can be found. Thus, we are led to the following theorem.

Theorem 3. *Consider the semi-infinite Toda field theory (1.1), with H and K given by (4.1) and (4.2), respectively, and with all rows of H different. Moreover, let HK have no zero columns. Then, the symmetry algebra depends on the fundamental spaces of the finite-dimensional submatrices $\tilde{H}$ and $\tilde{K}$, on the solutions of the difference equations (2.18) and (2.24) for $n \geqslant M + 1$ and, finally, on the solutions of the difference equation (2.21) for $m \geqslant N' + 1$.*

The theory is conformally invariant if the condition (4.8) holds. The corresponding generators take the form (2.25). Otherwise, if (4.8) does not hold, the symmetry reduces to the Poincaré group generated by (3.12).

A gauge transformation group, involving $2n_0$ arbitrary functions of one variable, exists if the relation (4.6) holds. The algebra generators take the form (4.7). Finally, $\hat{Z}$-type gauge transformations exist if not only (4.6) holds, but also the supplementary condition

$$N' - M' + q_2 + \dim\big(\mathrm{Ker}\big(\tilde{K}^T\big)\big) = m_0 > 0 \tag{4.10}$$

is satisfied. In such a case they form a Lie algebra of dimension $m_0 \times n_0$.

4.2. Special cases

Now let us consider the same three examples as in the previous sections.

4.2.1. Mikhailov–Fordy–Gibbons field theories. All examples of section 3.2 can be generalized to the semi-infinite case, simply allowing N to go to ∞ for each classical Lie algebra. The equations labelled by $1 \leqslant n \leqslant N_0$ are given explicitly by (3.15)–(3.17) and (3.19), respectively. Moreover, for $i \geqslant N_0$ the equations are the same as in the infinite case, i.e. equation (1.2).

For the $A_{\infty+}$ algebra (we use this notation in order to distinguish this semi-infinite model from the previously introduced A_∞ infinite one), we have $M = N = M' = N' = 0$ and hence the symmetries are exactly the same as in the infinite and in the finite cases (see equation (2.35)), where the summations are over the appropriate range.

For the B_∞ algebra one has $\tilde{H} = -\tilde{K} = (-1)$, then also $M = N = M' = N' = 1$, as one can see from (3.16). Theorem 3 allows one to establish that there are no gauge transformations of any kind and the generators of the conformal transformations are the same as given in (2.35).

From equation (3.17) one sees that $\tilde{H} = -\tilde{K} = (-2)$ for the C_∞ algebra, then $M = N = M' = N' = 1$. Thus, theorem 3 establishes that only the conformal invariance is admitted. Its generators have the same form as in equation (3.18), where the summation is over the positive integers.

Finally, for the D_∞ algebra one has

$$\tilde{H} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = -\tilde{K}^T.$$

Theorem 3 implies that only conformal transformations leave the system invariant and their generators are obtained by taking the limit $N \to \infty$ in the formulae (3.20).

4.2.2. The semi-infinite Toda field theory (1.3). The discussion is very simple. Indeed, since H is the identity matrix, there are no gauge transformations. Moreover, the generators of the conformal transformations in the infinite, semi-infinite and finite cases always take the same form (2.36), where the summations are over the appropriate range.

4.2.3. The semi-infinite Toda field theories (1.4). As opposed to the finite case, the matrix H is no longer invertible, so now these theories are not equivalent to those given by (1.3).

First, we observe that, since K is the identity matrix, there are no $\hat{Z}$-type transformations. For any classical Lie algebra, extended to $N \to \infty$, the recursive part of the systems, i.e. the equations labelled by $n \geqslant N_0$ as defined in section 3.2, are always the same as in the infinite case discussed in section 2.2.3. The solution of the corresponding difference equations for B_n ($n \geqslant N_0$), that is (2.18) and (2.24), are the same as in (2.38) and the generators are as in (2.39). However, for $1 \leqslant n < N_0$ the equations provide constraints of the form (4.3) and (4.4). The application of theorem 3 implies

(a) All the semi-infinite systems (1.4) are conformally invariant.
(b) All the semi-infinite systems (1.4) have $n_0 = 1$, as defined in (4.6), hence a gauge transformation algebra of the form (4.7) exists, with $j = 1$.

In the $A_{\infty+}$ case the $\hat{X}$ and $\hat{Y}$ conformal symmetries survive as in equation (2.39), and so do $\hat{U}_2$ and $\hat{V}_2$. However, the symmetries $\hat{U}_1$ and $\hat{V}_1$ are no longer present.

In the B_∞ case the generators $\hat{X}$, $\hat{Y}$ and $\hat{U}_2$, $\hat{V}_2$ combine together to give the new conformal symmetry generators,

$$\hat{X} = \xi(x)\partial_x + \tfrac{1}{2}\xi_x \sum_{n=1}^{\infty} n(n-1)\partial_{u_n} \qquad \hat{Y} = \eta(y)\partial_y + \tfrac{1}{2}\eta_y \sum_{n=1}^{\infty} n(n-1)\partial_{u_n}. \tag{4.11}$$

The remaining gauge invariance is generated by

$$\hat{U}(r) = r(x)\left[\partial_{u_1} + 2\sum_{n=2}^{\infty}\partial_{u_n}\right] \qquad \hat{V}(s) = s(y)\left[\partial_{u_1} + 2\sum_{n=2}^{\infty}\partial_{u_n}\right]. \tag{4.12}$$

For the C_∞ algebra the symmetry algebra is

$$\begin{aligned} \hat{X} &= \xi(x)\partial_x + \tfrac{1}{2}\xi_x \sum_{n=1}^{\infty} n(n-2)\partial_{u_n} \\ \hat{Y} &= \eta(y)\partial_y + \tfrac{1}{2}\eta_y \sum_{n=1}^{\infty} n(n-2)\partial_{u_n} \\ \hat{U}(r) &= r(x)\sum_{n=1}^{\infty}\partial_{u_n} \qquad \hat{V}(s) = s(y)\sum_{n=1}^{\infty}\partial_{u_n}. \end{aligned} \tag{4.13}$$

Finally, for the D_∞ algebra one has

$$\begin{aligned}
\hat{X} &= \xi(x)\partial_x + \tfrac{1}{2}\xi_x \sum_{n=1}^{\infty}(n-1)(n-2)\partial_{u_n} \\
\hat{Y} &= \eta(y)\partial_y + \tfrac{1}{2}\eta_y \sum_{n=1}^{\infty}(n-1)(n-2)\partial_{u_n} \\
\hat{U}(r) &= r(x)\left[\partial_{u_1} + \partial_{u_2} + 2\sum_{n=3}^{\infty}\partial_{u_n}\right] \\
\hat{V}(s) &= s(y)\left[\partial_{u_1} + \partial_{u_2} + 2\sum_{n=3}^{\infty}\partial_{u_n}\right].
\end{aligned} \tag{4.14}$$

The formulae for the semi-infinite models (1.4) are consistent with those obtained in the finite case in section 3.2.3. The generators of the conformal invariance, in each case, are simply obtainable by dropping all terms involving N. Conversely, the terms proportional to a power of N provide us with the gauge invariance generators in the semi-infinite extensions. In this limit, the functions $r = \xi_x$ and $s = \eta_y$ must be considered as new linearly independent functions.

5. Conclusions

We have introduced the generalized Toda system (1.1) and investigated its Lie point symmetry group. It turned out that in the infinite case ($-\infty < n < \infty$) these systems are always invariant under an infinite-dimensional group of conformal transformations. It is also gauge invariant, if a certain homogeneous linear difference equation (i.e. equation (2.18)) has non-trivial solutions. Further gauge transformations exist if another linear homogeneous difference equation (i.e. equation (2.21)) also has non-trivial solutions.

If we restrict the range of n to $1 \leqslant n < \infty$, in some cases the symmetry group remains the same, or is reduced to a subgroup of the original symmetry group. However, in other cases (see (4.12) and (4.14)) the symmetry group does not coincide with a Lie subgroup.

In the finite case, with $1 \leqslant n \leqslant N$, the symmetry group remains the same as in the semi-infinite case, or it is reduced further.

In some situations (see theorems 2 and 3) the infinite-dimensional conformal symmetry group is reduced to the Poincaré group in two dimensions (see equation (3.12)).

These results were obtained directly, that is by analysing the determining equations for the symmetries for all types of systems: infinite, semi-infinite and finite. The question to which we plan to devote a separate paper is the application of the infinite generalized Toda systems. In particular, we will establish the degree to which the symmetries of the semi-infinite and finite Toda systems are 'inherited' from those of the infinite systems. In other words, we plan to discuss symmetry breaking by boundary or periodicity conditions of the infinite chains.

One of the surprising results obtained in this paper is that the class of the conformally invariant Toda field theories is much larger than the class of the completely integrable models. Indeed, the existence of a Lax pair imposes severe algebraic restrictions on the matrices H and K (see, for instance, [19]).

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